Flow Around Simply and Multiply Connected Bodies: A New Iterative Scheme for Conformal Mapping

Paolo Luchini* and Fernando Manzo†
University of Naples, Naples, Italy

After discussing the techniques that can be adopted for numerical conformal mapping of single airfoils, cascades, and doubly connected domains, either for the purpose of calculating the irrotational flowfield around these configurations or of constructing a calculation grid for the numerical computation of more general types of flow, an iterative algorithm is presented that displays quadratic convergence. This algorithm, made possible by the analytical solution of the integral equation that arises from linearizing the problem, is several times faster than the classical Theodorsen-Garrick procedure under most conditions and can handle more general shapes for which the older method does not converge at all.

I. Introduction

THE numerical determination of a conformal mapping is a key step in several sectors of numerical fluid dynamics, in particular in its application to external flow. In fact, the knowledge of the conformal mapping of a general airfoil section onto a standard shape, typically a circle, allows the immediate determination of the inviscid, incompressible, steady flow around this airfoil, and the same transformation can be used to generate a suitable calculation grid for the numerical computation of viscous, compressible, or unsteady flowfields.[1-3]

Algorithms for the numerical calculation of conformal mappings have been developed for simply and multiply connected profiles, in particular doubly connected ones and periodic cascades.[1,4] The starting point for the development of all these algorithms has been Theodorsen and Garrick's iterative procedure[5-8] for transforming a near-circle into a circle. The required conformal mapping is generated through the successive application of an analytically determined mapping whose purpose is to transform the given profile into a cornerless near-circle and an iteratively obtained mapping of the near-circle onto a circle. Even if the calculation of this second mapping is nowadays implemented using fast Fourier transform (FFT) techniques, the basic iterative procedure is still the original one of Theodorsen and Garrick.

In this paper, an algorithm based on Newton iteration is presented and applied to simply and doubly connected bodies and to periodic cascades. This algorithm has two advantages: it has a faster convergence rate than Theodorsen and Garrick's, and it does not require the assumption that the profile to be transformed is a near-circle.

In most of our tests, the total runtime was less than that of Theodorsen and Garrick's algorithm, being about the same only when the profile was very close to a circle. When the precision required is high or the initial profile is not very close to a circle, the time advantage given by the present algorithm can be substantial. In addition, the present algorithm can find a transformation of profiles so different from a circle that the other one does not converge at all.

Received Aug. 3, 1987; revision received Jan. 21, 1988.

*Staff Scientist, Institute of Gasdynamics, Faculty of Engineering.
†Associate Professor, Institute of Gasdynamics, Faculty of Engineering.

While revising this paper, we learned that methods similar to ours already have been published in other mathematical journals without considering fluid dynamic applications. In particular, the basic Newton method for simply connected domains has been described by Wegmann, first in German in 1978[12] and later in 1986 in an English translation of the original paper.[11] The conditions under which quadratic convergence is obtained with respect to an appropriate norm in a Sobolev space were studied in Refs. 13 and 14.

Wegmann extended his method to doubly connected domains in 1986.[15] However, he only achieved quadratic convergence using twelve complex FFTs per step, although he also tested a modification using eight, lacking a theoretical justification. This worked but converged at a considerably slower rate than the other.

Our method for doubly connected domains, presented in Sec. VI, achieves quadratic convergence using eight complex FFTs per step and in the same test problem of Wegmann attains an error slightly smaller than his method using twelve FFTs per step, as illustrated in Table 1, thus affording a time reduction of a factor 2/3.

Two commendable reviews of modern methods for numerical conformal mapping, including Wegmann's, are presented in Refs. 16 and 17.

II. Calculation of Fluid Dynamic Fields by Conformal Mapping

There are two uses of conformal mapping in fluid dynamics. In the calculation of irrotational flowfields, conformal mapping of the studied domain onto a standard shape (a circle for single airfoils and cascades, an annulus for doubly connected domains) enables one to obtain the flowfield directly from the well-known solutions around the transformed shapes. In the calculation of more general types of flow, the same conformal mapping enables one to construct a body-fitted orthogonal grid for numerical computations, which allows both the equations and boundary conditions to be discretized at an advantage.

The numerical determination of a conformal mapping usually involves two steps. The first step is a conformal mapping expressed in analytical form, whose purpose is to eliminate corners in the initial shape and also to map, in the case of cascades, all the blades onto a single profile. A number of formulas to perform this pretransformation can be found in the literature.[1,4] The most time-consuming part of this step is not the application of the transformation itself but the determination of the free parameters that all those formulas contain. In fact, the standard method (due to Theodorsen and Garrick)

Table 1 Maximum correction effected at each iteration for Wegmann's algorithm (W) and the present one (P) for doubly connected domains

n	$\lvert\delta t_{1,n}\rvert_{max}$		$\lvert\delta t_{2,n}\rvert_{max}$		$r_{in+1} - r_i$	
	W	P	W	P	W	P
0	1.15	0.94	1.06	0.56	-0.11	-0.11
1	0.89	0.35	0.83	0.18	-0.078	-0.27
2	0.18	0.10	0.20	0.11	$-7.7\cdot10^{-3}$	$-6.2\cdot10^{-4}$
3	0.030	$3.1\cdot10^{-3}$	0.026	$5.1\cdot10^{-3}$	$3.0\cdot10^{-4}$	$-4.2\cdot10^{-6}$
4	$3.5\cdot10^{-4}$	$3.6\cdot10^{-5}$	$4.1\cdot10^{-5}$	$3.7\cdot10^{-5}$	$3.4\cdot10^{-7}$	$-3.1\cdot10^{-11}$
5	$5.7\cdot10^{-8}$	$3.2\cdot10^{-10}$	$2.5\cdot10^{-8}$	$4.6\cdot10^{-10}$	$-1.2\cdot10^{-9}$	$+2.1\cdot10^{-16}$
6	$8.9\cdot10^{-11}$	$3.1\cdot10^{-11}$	$3.0\cdot10^{-11}$	$1.3\cdot10^{-11}$	$-1.4\cdot10^{-16}$	$-2.1\cdot10^{-16}$
7	$6.7\cdot10^{-11}$	$2.7\cdot10^{-11}$	$2.3\cdot10^{-11}$	$1.9\cdot10^{-11}$	$1.4\cdot10^{-17}$	$-6.5\cdot10^{-16}$

used to implement the second numerical step requires a very careful choice of these parameters in such a manner that the intermediate shape obtained is as close as possible to a circle.

The second step is properly numerical, in the sense that it works on a discretized representation of the conformal mapping, and its purpose is to transform the intermediate cornerless shape into a circle (or an annulus in the case of a doubly connected domain). The intermediate shape usually is approximated by a cubic periodic spline fitted to a number of discrete points that have been subjected to the analytical pretransformation.

The next section contains a description of the pretransformations. The details of Theodorsen and Garrick's numerical method are given in Sec. IV, whereas those of the method presented in this paper are given in Sec. V. Let us only remark here that both methods use a Fourier series representation of the conformal mapping that can be rapidly obtained by the FFT algorithm.

III. Pretransformations

The standard procedure for calculating a conformal mapping, either to determine the irrotational flowfield around a given body configuration or to construct a computation grid for the numerical calculation of more general types of flow, consists in applying first one or more analytical pretransformations to round out the initial shape and then a numerical transformation to map the intermediate shape onto a circle.

Our procedure differs from the standard one only in the numerical transformation. Since, however, the pretransformations are needed for the complete calculation, let us describe them briefly in this section.

We shall need two kinds of pretransformation: one to eliminate corners in the initial profile and another, to be used in the case of periodic cascades, to transform an array of equal profiles into a single one. For the first purpose, one can use a Kármán-Trefftz transformation of the form

$$(Z - Z_1)/(Z - Z_2) = [(z - Z_1/k)/(z - Z_2/k)]^k \tag{1}$$

This transformation has two singular points Z_1 and Z_2, which are mapped to Z_1/k and Z_2/k and transform an airfoil profile given in the plane of the variable Z, running through one of the singular points, say Z_1, and forming there a corner of angle $(2-k)\pi$ into a smooth shape. The second singular point is arbitrary, provided it falls inside the given profile, and can be chosen so as to get the most convenient transformed shape.

For the purpose of transforming a rectilinear periodic array of blades into a single profile, the appropriate transformation is

$$2\pi i(Z - Z_\infty)/Z_p = \log[(z-1)/(z+1)] \tag{2a}$$

where Z_p is the period, i.e., the complex Z plane vector that joins corresponding points in one airfoil of the cascade and the next. The upstream and downstream points at infinity in the Z plane are carried by this transformation into $z = 1$ and $z = -1$, whereas the point Z_∞ is carried into infinity in the z plane. Just as Z_2 of Eq. (1), Z_∞ can be chosen so as to give the most convenient transformed shape, provided only that Z_∞ remains outside of the given profiles.

A similar formula can be written that transforms a circular array of blades into a single profile. This formula is

$$(Z - Z_0)/(Z_\infty - Z_0) = [(z-1)/(z+1)]^{1/n} \tag{2b}$$

where Z_0 is the center of the circular array and n the number of blades. The points $Z = Z_0$, ∞, and Z_∞ are mapped by this transformation to $z = 1$, -1, and ∞, respectively; Z_∞ is an adjustable free parameter just as in Eq. (2a).

It must be noted that Eqs. (1), (2a) and (2b) are probably the simplest formulas that can be used for pretransformation, and all three have a simple inverse. More complicated formulas can be found in the literature, in particular in connection to cascades and doubly connected domains, which contain more parameters and allow an intermediate shape closer to a circle to be obtained by a careful choice of these parameters. Since our method can work with shapes that are not close to a circle, the simpler equations, Eqs. (1), (2a), and (2b), are sufficient.

IV. Theodorsen-Garrick Algorithm

The purpose of the Theodorsen-Garrick algorithm is to transform a near-circle, obtained by suitable pretransformations, into a circle.

Let R, ϕ be polar coordinates in the Z plane and $R = R_0(\phi)$ the representation of a given curve sufficiently close to a circle (i.e., such that R_0 is sufficiently close to a constant). Let $z = re^{i\phi}$ be the complex variable in the transformed plane. The problem of conformal mapping is to find a complex function $Z(z)$, where $Z = Re^{i\phi}$, analytical for $|z| > 1$ and such that

$$Z(e^{i\theta}) = R_0(\phi)e^{i\phi} \tag{3}$$

and $Z(z) = O(z)$ when $z \to \infty$ [$O(z)$ being Landau's capital "O" symbol].

The function $Z(z)$ may be considered determined everywhere once it has been determined on the unit circle, $|z| = 1$, since then Cauchy's integral relation allows $Z(z)$ to be calculated for arbitrary values of its argument, as

$$\frac{Z(z)}{z^2} = \frac{1}{2\pi i}\oint \frac{Z(z')}{z'^2(z'-z)}\,dz' \tag{4}$$

the integral being extended to the circumference $|z'| = 1$. (It is necessary to apply Cauchy's formula to the analytical function Z/z^2 that, contrarily to Z, vanishes at infinity.) Since R is a known function of ϕ, $R_0(\phi)$, it is sufficient to determine the function $\phi(\theta)$ on the unit circle in the z plane to know the correspondence of the boundaries $Z(e^{i\theta})$.

On the other hand, if only the real (or imaginary) part of an analytic complex function $F(z)$, finite at infinity, is known on the unit circumference, Schwartz's formula allows the function itself to be determined up to an imaginary (or real) arbitrary constant as

$$F(z) = -\frac{1}{2\pi} \oint \frac{z' + z}{z' - z} \operatorname{Re}[F(z')] \, d\theta' \tag{5}$$

If $F(e^{i\theta})$ and its real part are represented by their Fourier series

$$F(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \quad \operatorname{Re}(F) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta} \tag{6}$$

Eq. (5) may be written as

$$a_n = [1 - \operatorname{sgn}(n)] b_n \tag{7}$$

where $\operatorname{sgn}(n) = (-1$ for $n<0$, 0 for $n=0$, 1 for $n>0)$. An arbitrary imaginary constant may be added to a_0.

In the numerical application of Schwartz's formula, the form of Eq. (7) is more convenient because the difficulty associated with the presence of a singularity in Eq. (5) is avoided and calculations can be considerably sped up by the use of the FFT algorithm.

The Theodorsen-Garrick algorithm is based on the repeated application of Eqs. (5) or (7), to the function $F(z) = \log(Z/z)$, which is analytical outside the unit circle and finite at infinity. The algorithm is as follows. Start with an approximation of the correspondence of the boundaries $\phi(\theta)$, say $\phi = \theta$. From the equation of the profile to be transformed, $R = R_0(\phi)$, calculate $R(\theta) = R_0[\phi(\theta)]$, and hence the real part of $\log(Z/z)$ on the circle $|z| = 1$, i.e., $\log(R) = \log\{R_0[\phi(\theta)]\}$. Insert this real part in Eq. (5) and obtain the imaginary part, $\operatorname{Im}[F(z)] = \phi - \theta$. Take $\phi(\theta) = \operatorname{Im}[F(z)] + \theta$ as a new approximation to $\phi(\theta)$ and iterate up to convergence.

This algorithm belongs to the class of the iterative procedures obtained by putting the equation to be solved in the form $x = f(x)$ and then calculating each iterate from the previous one as $x_{n+1} = f(x_n)$. If convergence is reached, in the sense that for some n the difference between x_{n+1} and x_n is negligible, the original equation has been solved. An example of an algorithm of this kind is the Jacobi iterative method for systems of linear equations. When the algorithm converges, the error of the $(n+1)$th iterate can be expected to be asymptotically a constant fraction of the error of the nth iterate.

The Theodorsen-Garrick method was proven to converge when the profile to be transformed is sufficiently close to a circle.[6] Its error, however, can only be expected to decrease by a constant factor, say α, at each iteration, so that after n iterations the error has decreased by α^n. This type of convergence is characterized as linear.

V. Quadratic Algorithm: Simply Connected Domain

Now we are going to present an iterative algorithm that transforms into a circle a smooth shape, not necessarily close to a circle, with quadratic convergence, i.e., with an error of the nth iterate asymptotically proportional to $\exp(-2^n)$.

A class of quadratic algorithms for solving iteratively nonlinear problems is the one obtained from the Newton-Raphson technique. This technique consists of deriving the correction to each iterate from the solution of a linearized problem, where the linearization parameter is the difference between the current iterate and the exact solution. Provided the linearized problem can be solved analytically, very fast algorithms are obtained in this way.

The Newton-Raphson technique may be applied to single equations, systems, and functional problems. In the case of conformal mapping, a functional problem, it may be formulated as follows. Let $t(\theta)$ be the unknown function and $N[t(\theta)] = 0$ the equation it must satisfy, N being a general nonlinear operator. If an approximation $t_n(\theta)$ to $t(\theta)$ is known, the nonlinear operator can be linearized with respect to the difference $\delta t(\theta) = t(\theta) - t_n(\theta)$, thereby transforming the original equation into $N[t_n(\theta)] + L[\delta t(\theta)] + O(\delta t^2) = 0$, where L, a linear operator, is the functional derivative or variation of the operator N with respect to δt. The error connected to this linearization, quadratic in δt, has been denoted by Landau's capital "O" symbol $O(\delta t^2)$. If the inhomogeneous linear equation $L[\delta t(\theta)] = f(\theta)$ can be solved—let us denote its solution by $\delta t = L^{-1}[f(\theta)]$—$\delta t$ may be determined as $\delta t(\theta) = -L^{-1}\{N[t_n(\theta)]\} + O(\delta t^2)$. Neglecting $O(\delta t^2)$ terms in this equation gives the next approximation $t_{n+1}(\theta) = t_n(\theta) - L^{-1}\{N[t_n(\theta)]\}$. The error associated with this procedure has the property that $\delta t_{n+1} = t - t_{n+1} = O(\delta t_n^2)$, so that one may expect the error to decrease asymptotically like the solution of the difference equation $r_{n+1} = \alpha r_n^2$, i.e., $\alpha r_n \simeq \exp(-2^n)$. Since second-order terms are neglected, this may be called a quadratic algorithm. With respect to first-order iterative algorithms, it has a biexponential rather than exponential convergence law.

The key step in applying the Newton-Raphson technique to functional problems is the analytical inversion of the linear operator $L[\delta t(\theta)]$. In the case of conformal mapping, this is a Riemann-Hilbert problem, which may be solved by two successive applications of Schwartz's formula according to the following procedure.

Let $X = X_0(t)$, $Y = Y_0(t)$ be the parametric representation, in the plane of the variable $Z = X + iY$, of the curve to be mapped onto the unit circle $|z| = 1$, and let $Z(z)$ be the mapping function where $z = re^{i\theta}$. Because of the possibility of applying Eqs. (4) or (5), the function $Z(z)$ may be considered known once the function $t(\theta)$ representing the correspondence of the boundaries is determined.

Our problem is then to find a function $t(\theta)$ such that $Z = Z_0[t(\theta)] = X_0[t(\theta)] + iY_0[t(\theta)]$ is an analytic function, i.e., obeys Eq. (5). According to the linearization procedure, we write $Z_0[t_n(\theta) + \delta t(\theta)] \simeq Z_0[t_n(\theta)] + (dZ_0/dt)\delta t(\theta)$. [Note that by dZ_0/dt we mean the derivative of the complex function $Z_0 = X_0(t) + iY_0(t)$ with respect to the real variable t. It is not a derivative in the complex plane.] We have thus the linear problem of finding a real function $\delta t(\theta)$ such that $Z_0[t_n(\theta)] + (dZ_0/dt)\delta t(\theta)$ is an analytic function.

In a first step, we determine an auxiliary analytic function $A(z)$ such that $\operatorname{Im}[A(z)] = \arg(dZ_0/dt) - \theta$ on the unit circle. This can be obtained from Eq. (5) as

$$A(z) = -\frac{i}{2\pi} \oint \frac{z' + z}{z' - z} \left[\arg\left(\frac{dZ_0}{dt}\right) - \theta' \right] d\theta' \tag{8}$$

Then, we make $Z_0[t_n(\theta)] + (dZ_0/dt)\delta t(\theta)$ an analytic function by requiring that its ratio by $z \exp(A)$ satisfy Eq. (5). In fact, if $A(z)$ is an analytic function, so is $\exp(A)$, and the ratio of two analytic functions is again an analytic function. The factor z is needed, as in the Theodorsen-Garrick method, to eliminate the divergence of $Z(z)$ at infinity. The auxiliary function has been determined so that $(dZ_0/dt)\delta t(\theta)/z \exp(A)$ is a real quantity (because its argument vanishes). Therefore, applying Eq. (5) to the function $-i\{Z_0[t_n(\theta)] + (dZ_0/dt)\delta t(\theta)\}/z \exp(A)$ gives

$$Z_0[t_n(\theta)] + \frac{dZ_0}{dt}\delta t(\theta) = -\frac{iz \exp(A)}{2\pi} \times \left[\oint \frac{z' + z}{z' - z} \operatorname{Im}\left(\frac{Z_0[t_n(\theta')]}{z' \exp[A(z')]}\right) d\theta' + b \right] \tag{9}$$

where the real constant b, which is arbitrary in Schwartz's formula, must be determined so that the ratio Z/z is real at infinity if rotation between the Z and z planes is to be avoided. The constant b may be easily calculated from the Fourier series representation of Schwartz's formula, since the limit for $z \to \infty$ of analytic functions like $A(z)$ and $Z/z \exp(A)$ equals the $n = 0$ term of their Fourier series expansion on the unit circle. In order to make the $n = 0$ coefficient of Z/z real, it is sufficient to calculate b so that the argument of the $n = 0$ coefficient of $Z/z \exp(A)$ is the same as the negative imaginary part of the $n = 0$ coefficient of A.

Equation (9) solves the problem of determining $\delta t(\theta)$. From the numerical point of view, this requires two applications of Schwartz's formula, Eqs. (5) or (7), each of which can be obtained by two successive FFTs. Once $\delta t(\theta)$ is calculated, the next approximation $t_{n+1}(\theta) = t_n(\theta) + \delta t(\theta)$ can be generated and the process repeated up to convergence.

VI. Quadratic Algorithm: Doubly Connected Domain

The previous algorithm can be extended to the transformation of a doubly connected domain, such as the space surrounding a pair of airfoils, into an annulus.

To this aim, we shall need the equivalent of Schwartz's formula for the annulus. This was found in Ref. 9 and was already discussed by the authors in Ref. 10. The result is the following.

Consider an annulus with external radius 1 and internal radius r_i. A function $F(z)$, where $z = r\exp(i\theta)$, analytical in this annulus, whose imaginary part equals $f_e(\theta)$ on the external circumference and $f_i(\theta)$ on the internal one, exists only if

$$\oint f_i \, d\theta = \oint f_e \, d\theta \tag{10}$$

In fact, the integral with respect to θ of the real or imaginary part of any function analytical in an annulus must be the same on the two circumferences.

Under the condition of Eq. (10), $F(z)$ may be expressed by its Laurent series

$$F = \sum_{n=-\infty}^{\infty} a_n z^n \tag{11}$$

where

$$a_n = \frac{(b_{en} - r_i^n b_{in})}{2(1 - r_i^{2n})} \quad \text{for} \quad n \neq 0 \tag{12a}$$

$$a_0 = b_{e0} = b_{i0} \tag{12b}$$

and b_{en} and b_{in} are the Fourier series coefficients of $f_e(\theta)$ and $f_i(\theta)$ [Eq. (10) implies that $b_{e0} = b_{i0}$].

In order to write an iterative formula for the conformal mapping problem, we shall, as in the previous section, apply Eq. (12) twice, determining the first time an auxiliary function.

Two airfoils can be carried by suitable pretransformations, e.g., two successive Kármán-Trefftz transformations, into two smooth curves one internal to the other. Let $X = X_1(t)$, $Y = Y_1(t)$ and $X = X_2(t)$, $Y = Y_2(t)$ be the parametric representations of these two closed curves in the plane of the complex variable $Z = X + iY$. The conformal mapping problem is to find two functions $t_1(\theta)$ and $t_2(\theta)$ such that a function $Z(z)$ exists analytical in the annulus and equal to $Z_1[t_1(\theta)] = X_1[t_1(\theta)] + iY_1[t_1(\theta)]$ on the external circumference and to $Z_2[t_2(\theta)] = X_2[t_2(\theta)] + iY_2[t_2(\theta)]$ on the internal one. This problem is well posed and has a unique solution (up to a rotation) only for a particular value of r_i.

The linearized form of the conformal mapping problem must simultaneously determine corrections to $t_1(\theta)$, $t_2(\theta)$, and r_i. Let r_{in} be the approximation of r_i obtained at the nth iteration. The value of the analytic function $Z(z)$ on the circumference $r = r_{in}$ may be written, with second-order accuracy, as $Z(r_{in}, \theta) = Z(r_i, \theta) - (dZ/dz)\delta r_i e^{i\theta} = Z_2[t_2(\theta)] - (dZ/dz)\delta r_i e^{i\theta}$, δr_i denoting the difference $r_i - r_{in}$. After linearizing, as before, the functions $Z_1(t)$ and $Z_2(t)$, we are left with the linear problem of determining an analytic function that equals $Z_1[t_{1n}(\theta)] + (dZ_1/dt)\delta t_1$ on the circumference $|z| = 1$ and $Z_2[t_{2n}(\theta)] + (dZ_2/dt)\delta t_2 - (dZ/dz)\delta r_i e^{i\theta}$ on the circumference $|z| = r_{in}$. The free constant δr_i must be determined so that the compatibility condition of Eq. (10) is satisfied and such a function exists.

To complete the statement of the problem, we must still indicate a suitable approximation of dZ/dz in terms of the nth iterate, since otherwise the problem would not be linear. Let us postpone for a moment the choice of this approximation; we only want to remark here that a first-order approximation is sufficient, because dZ/dz is multiplied by the first-order quantity δr_i and a first-order error in dZ/dz gives rise to a second-order error in Z that is tolerable. By the same token, a first-order change in dZ_1/dt and/or dZ_2/dt, whose purpose will be seen next, may be tolerated.

Just as in the case of the simply connected domain, the linearized problem is a Riemann-Hilbert problem and may be solved by two successive applications of Eqs. (11) and (12). We first determine an auxiliary analytic function $A(z)$ such that $\mathrm{Im}[A(z)]$ equals $\arg(dZ_1/dt) - \theta + c$ for $|z| = 1$ and $\arg(dZ_2/dt) - \theta - c$ for $|z| = r_{in}$, by calculating the constant c so that Eq. (10) is satisfied [this may be simply done after the Fourier transformation by calculating the coefficient a_o as $(b_{eo} + b_{io})/2$]. Notice that $\arg(dZ_{1,2}/dt) = \arg(dZ_{1,2}/d\theta)$, because $dZ_{1,2}/d\theta = (dZ_{1,2}/dt)(dt/d\theta)$ and $dt/d\theta$ is real, and at convergence $dZ_{1,2}/d\theta = dZ/d\theta = iz\, dZ/dz$ is an analytic function. Therefore, $\arg(dZ_{1,2}/dt) - \theta$ differs only at first order from the argument of an analytic function, which obeys Eq. (10) automatically, and c is a first-order small quantity. We may then replace $dZ_{1,2}/dt$ by $e^{\pm ic}\, dZ_{1,2}/dt$ in the statement of the problem and seek an analytic function $F \simeq Z/z \exp(A)$ such that $\mathrm{Im}(F)$ equals $\mathrm{Im}(Z_1/e^{i\theta} \exp A)$ for $|z| = 1$ and $\mathrm{Im}\{[Z_2 - (dZ/dz)\delta r_i e^{i\theta}]/r_{in} e^{i\theta} \exp A\}$ for $|z| = r_{in}$, after calculating δr_i so that Eq. (10) is satisfied. After this second application of Eqs. (11) and (12), it is possible to obtain δt_1 and δt_2 from

$$Re\left\{\frac{Z_1[t_{1n}(\theta)]}{z \exp A}\right\} + \frac{e^{ic}\, dZ_1/dt}{z \exp A}\, \delta t_1 = Re[F(e^{i\theta})] \tag{13}$$

$$Re\left\{\frac{Z_2[t_{2n}(\theta)] - (dZ/dz)\delta r_i e^{i\theta}}{z \exp A}\right\} + \frac{e^{-ic}\, dZ_2/dt}{z \exp A}\, \delta t_2 = Re[F(r_{in} e^{i\theta})] \tag{14}$$

where the coefficients of δt_1 and δt_2 are real by construction. Then, the next approximation $t_{1,n+1} = t_{1,n} + \delta t_1$, $t_{2,n+1} = t_{2,n} + \delta t_2$ and $r_{i,n+1} = r_{in} + \delta r_i$ may be generated and the process iterated.

We have left still unanswered the question of how to determine a suitable first-order approximation of dZ/dz. There are two ways. One is to write dZ/dz as $(iz)^{-1}\, dz/d\theta = (iz)^{-1}(dZ/dt)(dt/d\theta)$, then replace dZ/dt by dZ_2/dt evaluated at the nth iteration, which may be done within a first-order approximation, and $dt/d\theta$ by a difference expression based on the discrete representation of $t_{2n}(\theta)$. This procedure works but is slightly unsatisfactory because it introduces a difference approximation that may not have the same accuracy as the rest of the calculation and might slow down the convergence. A second possibility is to exploit the observation that $z \exp A$, being defined as an analytic function whose argument is $\arg(dZ_{1,2}/dt) + c = \arg(dZ_{1,2}/d\theta) \pm c$, becomes coincident at convergence with $dZ/d\theta = iz\, dZ/dz$ up to a real multiplicative constant. Therefore, $-iC \exp A$ may be used as a first-order approximation to dZ/dz if only the constant C can be determined. For instance, C may be calculated from the condition that the integral over the whole circumference of $dt/d\theta = (dZ/d\theta)/(dZ/dt) = Cz \exp A/(dZ/dt)$ must equal the known increment of the parameter t after a full turn; $(dZ/dz)\delta r_i e^{i\theta}/z \exp A$ then may be replaced by $-iC\delta r_i/r_{in}$ and disappears from Eq. (14).

To summarize, after obtaining the auxiliary function A, one must first calculate the constant C as the given increment of the parameter t in a full turn, divided by the integral over the inner circumference of $z \exp A/(dZ_2/dt)$; then determine, with the aid of Eqs. (11) and (12), the analytic function F whose imaginary part equals $\mathrm{Im}(Z_1/z \exp A)$ and $\mathrm{Im}(Z_2/z \exp A) + C\delta r_i/r_{in}$, respectively, on the two circumferences, choosing δr_i so that Eq. (10) is satisfied [this can be simply done by choosing $a_0 = b_{e0}$ in Eq. (12) and then setting $\delta r_i = r_{in}(b_{e0} - b_{i0})/C$]. Finally, calculate δt_1 and δt_2 from Eqs. (13) and (14), where the term $(\mathrm{d}Z/\mathrm{d}z)\delta r_i e^{i\theta}$ may be omitted altogether.

In contrast to ours, the algorithm developed by Wegmann for doubly connected regions[15] does not initially linearize the problem with respect to δr_i, whereas it does so with respect to $\delta t_{1,2}$ and undertakes to calculate, at every iteration, suitable corrections $\delta t_{1,2}$ and a new radius $r_{i,n+1}$ such that there exists an analytic function that equals $Z_1[t_{1n}(\theta)] + (\mathrm{d}Z_1/\mathrm{d}t)\delta t_1$ for $|z| = 1$ and $Z_2[t_{2n}(\theta)] + (\mathrm{d}Z_2/\mathrm{d}t)\delta t_2$ for $|z| = r_{i,n+1}$. At a second time, the equation thus obtained for $r_{i,n+1}$ is actually linearized and a correction $\delta r_i - r_{i,n+1} - r_{i,n}$ is calculated by the Newton method, but having formulated the problem for $\delta t_{1,2}$ on the circumference of radius $r_{i,n+1}$ obliges one to calculate the Laurent coefficients of the analytic functions A and F twice in each iteration, which cannot be done with less than twelve FFTs. Simply omitting the recalculation of A and F, as tried by Wegmann in order to reduce the time required for each iteration, is not quite a satisfactory solution, as observed by Wegmann himself.

Instead, formulating the problem on the known circumference $|z| = r_{in}$ from the start gives a proper Newton iteration that requires only eight FFTs per step.

VII. Application to a Single Airfoil

From a theoretical point of view, the algorithm presented has a faster (quadratic) type of convergence than that of Theodorsen and Garrick. In order to compare in practice the time requirements of the two methods, we performed a number of tests.

One of the merits of this method with respect to Theodorsen and Garrick's is its relative insensitivity to the closeness of the initial profile to a circle. To test this insensitivity, we applied both methods to the transformation of ellipses of a different aspect ratio.

As a description of the profile, Theodorsen and Garrick's algorithm uses the polar representation $R = R_0(\phi)$, where R and ϕ are polar coordinates in the Z plane, whereas our algorithm uses a general Cartesian parametric representation $X = X_0(t)$, $Y = Y_0(t)$. For the sake of uniformity, the angle ϕ was used as parameter t. As an initial guess, $\phi(\theta) = \theta$ was used for both programs.

Tests were run with both algorithms employing a 256-point FFT and stopping the iteration when the maximum of $\delta\phi(\theta)$ was less than 10^{-5}, $\delta\phi(\theta)$ being the difference of $\phi(\theta)$ between successive iterates. The number of iterations and the time in seconds taken by the two algorithms to complete their task on an HP9826 computer are reported in Table 2.

The table shows that the present algorithm, contrary to Theodorsen and Garrick's, is insensitive to the proximity of the initial shape to a circle. In fact, at an aspect ratio of 1.2, the required time is approximately the same for the two programs, but with an increasing aspect ratio the present algorithm does not vary its time appreciably, whereas the other one takes a very rapidly increasing time and eventually, for values of the aspect ratio greater than 2, does not converge at all.

Table 2 Time requirements of the two algorithms for ellipses of different aspect ratios

	Theodorsen and Garrick		New algorithm	
Aspect ratio	Steps	Time	Steps	Time
1.2	6	120	3	128
1.5	11	221	3	128
2.0	31	626	4	171
3.0	N.C.		4	171
5.0	N.C.		4	171

In order to compare the performance of the two methods in a practically significant problem, a conformal mapping of the two airfoils NACA 4418 and NACA 6599A_{10}18 was computed. In these cases, a Kármán-Trefftz pretransformation of the form [Eq. (1)] had to be used to eliminate the trailing-edge corner. Different intermediate shapes, more or less close to a circle, were obtained for different choices of the position of the second singular point of Eq. (1), Z_2.

As is usually done, the profile, initially given at discrete points, was interpolated by means of a periodic cubic spline after the pretransformation. Although with Theodorsen and Garrick's algorithm a spline was used to interpolate the function $R_0(\phi)$, for the present algorithm an approximation of the curvilinear abscissa s was first calculated by summing up the relative distances between the given discrete points and then two independent splines were fitted to $X(s)$ and $Y(s)$. As an initial guess to start up the iteration, $s \sim \theta$ was used. Tests were run with both algorithms using a 256-point FFT and stopping the iteration when the maximum of $\delta\phi(\theta)$ was less than 10^{-4}.

Table 3 shows the number of iterations and time taken by the two algorithms to find a conformal mapping of the NACA 4418 airfoil for different choices of the abscissa X_2 of the second

Table 3 Time requirements for the NACA 4418 airfoil subjected to different pretransformation

	Theodorsen and Garrick		New algorithm	
X_2	Steps	Time	Steps	Time
−0.485	6	120	3	126
−0.45	18	361	4	168
−0.4	N.C.		4	168
0	N.C.		4	168

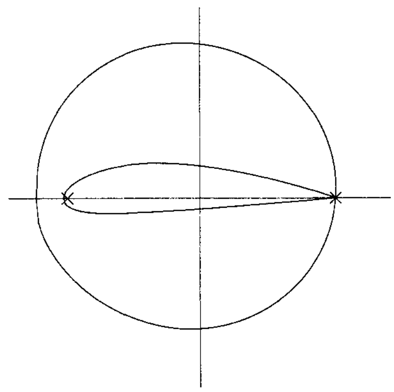

Fig. 1 Kármán-Trefftz transformation of the NACA 4418 airfoil. The locations of the two singular points, depicted by crosses, are $Z_1 = 0.5$ and $Z_2 = -0.485$.

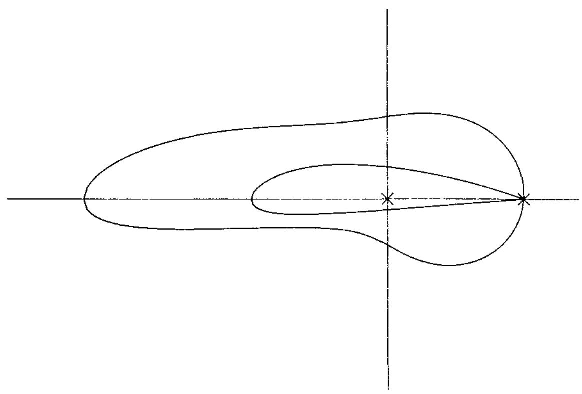

Fig. 2 Kármán-Trefftz transformation of the NACA 4418 airfoil for $Z_2 = 0$.

singular point Z_2 of the Kármán-Trefftz pretransformation. At $X_2 = -0.485$, when the intermediate shape is very close to a circle, as shown in Fig. 1, the runtime is about the same; as soon as X_2 differs even slightly from this value, however, Theodorsen and Garrick's algorithm takes a considerably longer time and already at $X_2 = -0.4$ no longer converges. The present algorithm, on the other hand, works without an important time degradation even at $X_2 = 0$, when the intermediate shape, shown in Fig. 2, is anything but a near-circle.

Therefore, this algorithm makes the fine-tuning of the Kármán-Trefftz pretransformation unnecessary, saving the human time required for this operation.

Similar timing data for the NACA 6599A$_{10}$18 airfoil are given in Table 4. In this case, having a smaller leading-edge curvature radius, Theodorsen and Garrick's algorithm does not converge already for $X_2 > -0.46$. (In both cases, the position of the leading edge is at $X_2 = -0.5$.)

Finally, the streamlines of the irrotational flowfield around the the two tested airfoils are given in Figs. 3 and 4.

VIII. Application to Cascades

If we use the pretransformation given by Eqs. (2a) or (2b), the region outside a rectilinear or a circular cascade of equal airfoils can be mapped onto a simply connected domain. If the airfoils forming the cascade have a sharp corner at their trailing edge, the transformed shape will have a similar corner, which can be successively eliminated by a Kármán-Trefftz transformation.

In this way, the problem is reduced, just as in the case of a single airfoil, to the transformation of a smooth shape into a circle and can be solved by the algorithm of Sec. IV. This final step presents no difference with respect to the case of a single airfoil and all the considerations of the previous section apply.

As an example, we considered a rectilinear cascade of NACA 4418 airfoils with a blade spacing of twice the chord. Figure 5 shows the initial profile, the first intermediate, obtained after a transformation using Eq. (2) with a point Z_∞ located half-way between two successive blades, and the second intermediate, obtained after a Kármán-Trefftz transformation with Eq. (1). A streamline pattern for this cascade is shown in Fig. 6.

IX. Application to a Doubly Connected Domain

For the conformal mapping of a pair of airfoils, the present method allows a simpler pretransformation to be used to eliminate corners. In fact, since it is not necessary that the intermediate shapes be very close to circles, it is sufficient to employ two independent Kármán-Trefftz transformations of the form [Eq. (1)] rather than the more complicated ones adopted in Ref. 1 in order to modify one airfoil at a time while leaving a certain circle invariant. The runtime requirements of the new algorithm for doubly connected domains were generally consistent in our tests with those of the single-profile version.

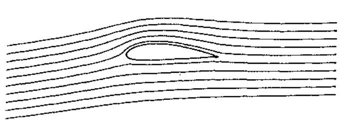

Fig. 3 Streamlines around the NACA 4418 airfoil for an attack angle of 6 deg.

Table 4 Time requirements for the NACA 6599A$_{10}$18 airfoil

X_2	Theodorsen and Garrick		New algorithm	
	Steps	Time	Steps	Time
−0.49	8	161	4	168
−0.47	25	503	4	168
−0.46	N.C.		4	168
0	N.C.		4	168

An example is presented in Fig. 7, where the conformal mapping of a certain given doubly connected domain is evidenced by a polar grid in the annulus together with its transformation. In order to calculate this particular example, five iterations were necessary using a 128-point FFT.

To compare our version of the Newton method for doubly connected domains, which uses eight FFTs per step, with Wegmann's,[15] which uses twelve, we applied it to the same test case considered in Ref. 15, two circles with radius 1 and 0.6, respectively, having their centers offset by 0.3 with respect to each other (the conformal mapping for this case is analytically known), using the same initial guess ($t_1 = \theta$, $t_2 = \theta + 0.2$,

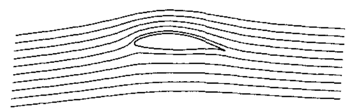

Fig. 4 Streamlines around the NACA 6599A$_{10}$18 airfoil for an attack angle of 6 deg.

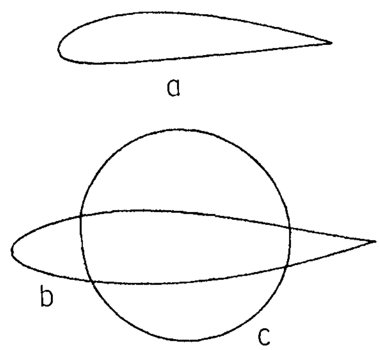

Fig. 5 Pretransformation of a rectilinear cascade; a) blade profile (NACA 4418); b) intermediate shape after transformation [Eq. (2)]; and c) second intermediate after transformation [Eq. (1)].

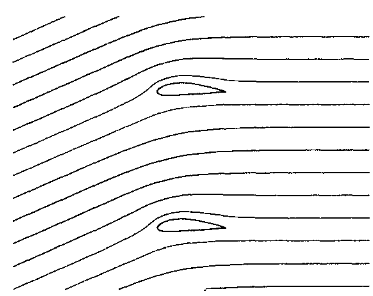

Fig. 6 Streamlines around a cascade of NACA 4418 airfoils.

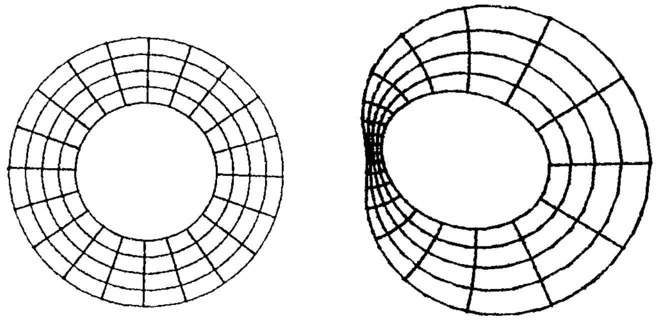

Fig. 7 Transformation of a polar grid in a given doubly connected domain.

$r_{io} = 0.6$) as Wegmann and the same number of points in the FFT (64). Table 4 reports the maximum calculated values of $\delta t_{1,n}$ and $\delta t_{2,n}$ at each iteration and the difference between the inner radius $r_{i,n+1}$ calculated at the nth iteration and the exact value of r_i for Wegmann's algorithm (W) and the present (P) (Wegmann's values are taken from Table 1 of Ref. 15). As may be seen, the present algorithm has an error comparable to (actually, slightly smaller than) Wegmann's for the same number of steps. The time required for every step, however, is reduced by a factor 2/3.

X. Conclusions

A quadratic iterative algorithm for conformal mapping has been described that can usefully replace the traditional Theodorsen-Garrick one in fluid dynamic numerical calculations for single airfoils, cascades, and doubly connected domains.

The present algorithm requires an only slightly longer code than Theodorsen and Garrick's and about the same data memory occupation. The execution time for each iteration step is approximately doubled because two applications of Schwartz's formula are needed rather than one, but this is more than compensated for by the faster convergence rate.

In addition, whereas Theodorsen and Garrick's method was explicitly conceived for a near-circle, represented in a polar coordinate as $R = R_0(\phi)$, near-circles play no role in the present algorithm, which uses a more general parametric representation $X = X_0(t)$, $Y = Y_0(t)$ of the profile and also can handle curves not admitting a polar representation. Therefore, the new algorithm can treat directly more complicated shapes than the previous one and allows for simpler pretransformations.

In practice, the method turned out to be quite insensitive to the proximity of the initial profile to a circle, taking about the same time to run in all situations. Theodorsen and Garrick's runtime, on the other hand, became much longer as soon as the initial profile was not very close to a circle; when the difference from a circle became significant, Theodorsen and Garrick's method ceased to converge at all, while ours still worked in about the same time as before.

When the shape becomes less and less regular, the convergence of the present algorithm is eventually limited by the discretization errors of the FFT. This is already seen in the case of the ellipse, in which with a 256-point FFT the performance of the algorithm degradates at an aspect ratio of about 7, but this value is easily exceeded using 512 or more points. In all our tests, we have not been able to identify any case in which failure of the algorithm could not be ascribed to discretization errors.

In fact, the uniform discretization imposed by the FFT poses some serious limitations onto all the currently used conformal mapping algorithms when one is interested in shapes having local details on small scale. This difficulty arises, for instance, when dealing with closely spaced cascades, because the pretransformation maps most of the airfoil surface onto a very small portion of the intermediate shape. For these cases, an algorithm that does not make use of Fourier transforms ought to be developed.

As explained in the introduction, a method similar to ours has been recently discussed in other mathematical journals, giving formal proofs of convergence but without considering fluid dynamic applications. Although in the case of simply connected domains, this method, due to Wegmann, is based on the same principles as ours, in the case of doubly connected domains, we have achieved a simplification in the recalculation of the radius of the inner circle by transferring the integral equation from the unknown to a known circle before each step. This simplification produced a time reduction of a factor 2/3, making it possible to perform an iteration step using eight FFTs instead of Wegmann's twelve.

Acknowledgment

This research was supported by the Italian Ministry of Public Education.

References

[1]Ives, D. C., "A Modern Look at Conformal Mapping Including Multiply Connected Regions," *AIAA Journal*, Vol. 14, Aug. 1976, pp. 1006–1011.

[2]Moretti, G., "Grid Generation Using Classical Techniques," Workshop on Grid Generation, NASA Langley, Oct. 1980.

[3]Jameson, A., "Iterative Solution of Transonic Flows Over Airfoils and Wings, Including Flows at Mach 1," *Communications on Pure and Applied Mathematics*, Vol. 27, May 1974, pp. 283–309.

[4]Zannetti, L. and Tamiru Ayele, T., "Incompressible Potential Flow Past Blades in Cascade: Classical Techniques and Modern Computing Instruments," *Proceedings of the Eighth AIDAA Conference*, Associazione Italiana di Aeronautica e Astronautica, Torino, Italy, 1986, pp. 43–58.

[5]Theodorsen, T. and Garrick, I. E., "General Potential Theory of Arbitrary Wing Sections," NACA TR-452, 1933.

[6]Warschawsky, S. E., "On Theodorsen's Method of Conformal Mapping of Nearly Circular Regions," *Quarterly Journal of Applied Mathematics*, Vol. 3, No. 2, April 1945, pp. 12–28.

[7]Garrick, I. E., "Potential Flow About Arbitrary Biplane Wing Sections," NACA TR-542, 1936.

[8]Garrick, I. E., "On the Plane Potential Flow Past a Lattice of Arbitrary Airfoils," NACA TR-788, 1944.

[9]Lagally, M., "Die reisbrungslose Strömung in Aussengebiete zwier Kriese," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 9, No. 4, Aug. 1929, pp. 299–305.

[10]Manzo, F. and Luchini, P., "Trasformazione Conforme 'One Step' di un Dominio Biconnesso in una Corona Circolare," *Proceedings of the Eighth AIMETA Conference*, Associazione Italiana di Meccanica Teorica ed Applicata, Torino, Italy, 1986, pp. 737–742.

[11]Wegmann, R., "An Iterative Method for Conformal Mapping," *Journal of Computational Applied Mathematics*, Vol. 14, Jan. 1986, pp. 7–18.

[12]Wegmann, R., "Ein Iterationsverfahren zur konformen Abbildung," *Numerische Mathematik*, Vol. 30, No. 3, June 1978, pp. 453–466.

[13]Wegmann, R., "Convergence Proofs and Error Estimates for an Iterative Method for Conformal Mapping," *Numerical Mathematics*, Vol. 44, March 1984, pp. 453–461.

[14]Hübner, O., "The Newton Method for Solving the Theodorsen Integral Equation," *Journal of Computational Applied Mathematics*, Vol. 14, Jan. 1986, pp. 19–30.

[15]Wegmann, R., "An Iterative Method for the Conformal Mapping of Doubly Connected Regions," *Journal of Computational Applied Mathematics*, Vol. 14, Jan. 1986, pp. 79–98.

[16]Henrici, P., *Applied and Computational Complex Analysis*, Vol. III, Wiley, New York, 1986.

[17]*Numerical Conformal Mapping*, edited by L. N. Trefethen, North-Holland, Amsterdam, the Netherlands, 1986.